Original scientific paper
Accepted 06. 04. 2001.

RADIMIR VIHER

The Consequences of Descartes's Method for Factorization of 4th Degree Polynomial

Posljedice Descartesove metode za faktorizaciju polinoma 4. stupnja

SAŽETAK

U članku je dan detaljan opis Descartesove metode za faktorizaciju polinoma četvrtog stupnja (nad poljem $\mathbf{R}$) koji je dan u sljedećem reduciranom obliku

$$P_4(x) \equiv x^4 + a_2x^2 + a_1x + a_0 \equiv (x^2 + Ax + B)(x^2 + Cx + D).$$

Nakon što je riješen sustav od četiri jednadžbe sa četiri nepoznanice, koji slijedi iz gornjeg identiteta, dobiva se sljedeća kubna rezolventa $P_3(t) \equiv t^3 + 2a_2t^2 + (a_2^2 - 4a_0)t - a_1^2$, gdje je $t = A^2$. Formulirana su i dokazana dva teorema. U prvom se otkriva korespondencija između tipova korijena od $P_3(t)$ i od $P_4(x)$ dok se u drugom daje karakterizacija tih tipova korijena od $P_3(t)$.

Ključne riječi: Descartesova metoda, faktorizacija, kubna rezolventa, tipovi korijena, karakterizacija tipova korijena, ravninske krivulje četvrtog reda

The Consequences of Descartes's Method for Factorization of 4th Degree Polynomial

ABSTRACT

In this article we give in details description of Descartes's method for factorization of the fourth degree polynomial (over the field $\mathbf{R}$) in the following reduced form

$$P_4(x) \equiv x^4 + a_2x^2 + a_1x + a_0 \equiv (x^2 + Ax + B)(x^2 + Cx + D).$$

When we seek the solution for A we get the following cubic resolvent $P_3(t) \equiv t^3 + 2a_2t^2 + (a_2^2 - 4a_0)t - a_1^2$, where $t = A^2$. At the end, we formulate and prove two theorems. In the first one, we find the correspondences between the types of the roots of $P_3(t)$ and $P_4(x)$ while in the second one, we give the characterizations of types of roots for $P_3(t)$.

Key words: Descartes's method, factorization, cubic resolvent, types of roots, characterizations of types of roots, plane quartic curves

MSC 2000: 14H45

If we can find the roots of the equation

$$P_4(x) \equiv a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 = 0$$
$$a_i \in \mathbf{R},\ i = 1, 2, 3, 4;\ a_4 \neq 0, \tag{1}$$

then we can also solve the problem of factorization of the polynomial $P_4(x)$ over the field $\mathbf{R}$. The history of the mathematics (see [1]) knows two basic methods for finding the roots of (1) the Ferrari's and Euler's method. However, the inverse reasoning is also true, if we know the factorization of $P_4(x)$ over $\mathbf{R}$ then we can also find all roots easily by solving two quadratic equations.

In the Descartes's method (see [5]) for factorization of $P_4(x)$ we can first suppose without losing generality that $a_4 = 1$ and that $a_3 = 0$. It is known that if $a_3 \neq 0$ then by supstituting $x = y - a_3/4$ we get $a_3' = 0$. Now we shall describe the method for factorization of the polynomial

$$P_4(x) \equiv x^4 + a_2x^2 + a_1x + a_0 \equiv (x^2 + Ax + B)(x^2 + Cx + D), \tag{2}$$

over $\mathbf{R}$.

From (2) we get the following system of nonlinear equations

$$\begin{aligned} A + C &= 0 \\ B + D + AC &= a_2 \\ AD + BC &= a_1 \\ BD &= a_0. \end{aligned} \tag{3}$$

When we substitute $C = -A$ in the second and the third equation we get

$$\begin{aligned} B + D &= A^2 + a_2 \\ -B + D &= a_1/A \\ BD &= a_0. \end{aligned} \tag{4}$$

Let us suppose that $A \neq 0$, then from the first and the second equation we get easily

$$\begin{aligned} B &= \frac{A^2 + a_2}{2} - \frac{a_1}{2A} \\ D &= \frac{A^2 + a_2}{2} + \frac{a_1}{2A}. \end{aligned} \tag{5}$$

Finally, from (5) and from the last equation in (4) it follows

$$A^6+2a_2A^4+(a_2^2-4a_0)A^2-a_1^2=0. \qquad (6)$$

After substituting $t=A^2$ in (6), we get the following cubic equation (resolvent)

$$t^3+2a_2t^2+(a_2^2-4a_0)t-a_1^2=0. \qquad (7)$$

Denote by $P_3(t)$ the left side of (7). Since $\lim_{t\to+\infty}P_3(t)=+\infty$ and $P_3(0)=-a_1^2$, when $a_1\neq 0$, it follows that there is a positive root of (7). Hence, there is a real root of (6) that is different from zero, and we can calculate B and D using the formulas (5). There is only the case $a_1=0$ left to be examined. Then we get the following system from (3)

$$\begin{aligned} A+C&=0\\ B+D+AC&=a_2\\ AD+BC&=0\\ BD&=a_0. \end{aligned} \qquad (8)$$

Analogously we get from (8)

$$\begin{aligned} B+D&=A^2+a_2\\ A(D-B)&=0\\ BD&=a_0. \end{aligned} \qquad (9)$$

From the second equation in (9) we get $A=0$ or $D=B$. If $A=0$ we get from (9)

$$\begin{aligned} B+D&=a_2\\ BD&=a_0. \end{aligned} \qquad (10)$$

From (10) we get

$$\begin{aligned} B&=\frac{a_2}{2}+\sqrt{\frac{a_2^2}{4}-a_0}\\ D&=\frac{a_2}{2}-\sqrt{\frac{a_2^2}{4}-a_0}, \end{aligned} \qquad (11)$$

and if $a_2^2-4a_0\geq 0$ we have the complete solution of (8). But when $a_2^2-4a_0<0$, then evidently $a_0>0$ and we must apply the second case $D=B$, which yields

$$\begin{aligned} D=B&=\sqrt{a_0}\\ A&=\sqrt{2\sqrt{a_0}-a_2}. \end{aligned} \qquad (12)$$

Thus, we always have a nonnegative root of the equation (6) and a complete solution of the system (3) in real numbers. There is only the equation (7) left to be examined. Using the known substitution (see [4]) $t=z-2a_2/3$ we get from (7)

$$z^3+pz+q=0, \qquad (13)$$

and the connection between p, q, a_0, a_1, a_2 is

$$\begin{aligned} p&=-4a_0-\frac{1}{3}a_2^2\\ q&=\frac{8}{3}a_0a_2-a_1^2-\frac{2}{27}a_2^3. \end{aligned} \qquad (14)$$

Let $a_1\neq 0$ and

$$D_1=\frac{q^2}{4}+\frac{p^3}{27}. \qquad (15)$$

It is known (see [3]) if $D_1>0$ then

$$z=\sqrt[3]{-\frac{q}{2}+\sqrt{D_1}}+\sqrt[3]{-\frac{q}{2}-\sqrt{D_1}} \qquad (16)$$

is only a real root of (13), so that

$$t=\sqrt[3]{-\frac{q}{2}+\sqrt{D_1}}+\sqrt[3]{-\frac{q}{2}-\sqrt{D_1}}-\frac{2a_2}{3} \qquad (17)$$

is surely a positive root of (7). If $D_1=0$

$$z=\begin{cases} 2\sqrt[3]{-\frac{q}{2}} & \text{if } q<0\\ \sqrt[3]{\frac{q}{2}} & \text{if } q>0\\ 0 & \text{if } q=0 \end{cases} \qquad (18)$$

is maximal nonnegative root of (13) (see [4]), so that

$$t=z-\frac{2a_2}{3} \qquad (19)$$

is surely a positive root of (7). If $D_1<0$, it can be shown, that if

$$\varphi=\arccos\left[-\frac{q}{2}\left(-\frac{3}{p}\right)^{\frac{3}{2}}\right], \qquad (20)$$

then maximal real root of (13) is (see [4])

$$z=2\sqrt{-\frac{p}{3}}\cos\frac{\varphi}{3}. \qquad (21)$$

Hence,

$$t=2\sqrt{-\frac{p}{3}}\cos\frac{\varphi}{3}-\frac{2a_2}{3} \qquad (22)$$

is a positive root of (7). In all these cases

$$A=\sqrt{t}, \qquad (23)$$

is a positive root of (6) and the other values of unknowns of the system (3) are given by (5).

Now we shall formulate and prove the theorem that gives the correspondences between the types of roots of $P_4(x)$ and the types of roots of its cubic resolvent $P_3(t)$ and a theorem that gives the characterizations for the types of roots of $P_3(t)$. We have three main possibilities for the types of roots of $P_3(t)$ (see Figure 1.).

In the **first case**, $P_3(t)$ has only one real nonnegative root and two conjugate complex roots or one real nonnegative root and one real negative double root.

In the **second case**, $P_3(t)$ has one real nonnegative root and two different real nonpositive roots.

In the **third case**, $P_3(t)$ has three real nonnegative roots (the cases of double and triple roots are included in this case).

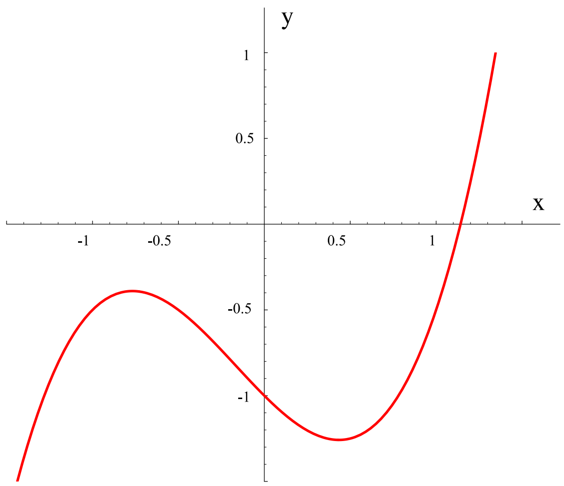

Fig. 1a: 1st case

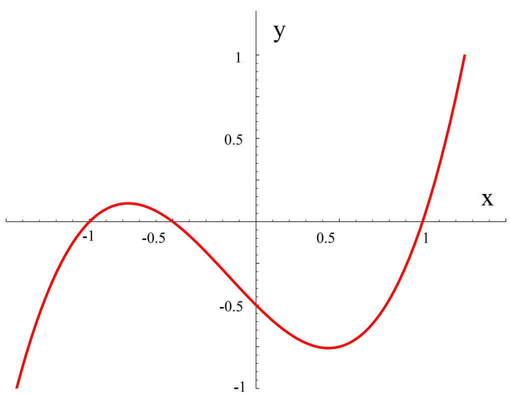

Fig. 1b: 2nd case

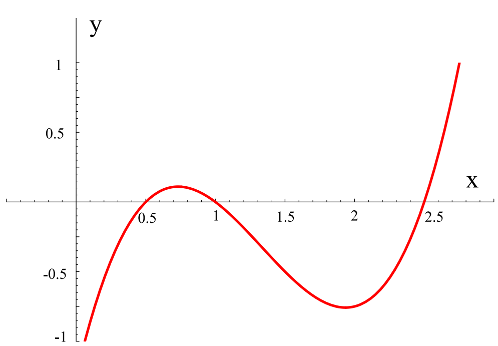

Fig. 1c: 3rd case

Theorem 1.

lst case $\Longleftrightarrow$ $P_4(x)$ has two real and two complex roots

2nd case $\Longleftrightarrow$ $P_4(x)$ has only complex roots

3rd case $\Longleftrightarrow$ $P_4(x)$ has only real roots

Proof. Let $P_4(x)$ have two real and two complex roots

$$\begin{aligned} P_4(x) &= (x-x_1)(x-x_2)(x-a-b\,i)(x-a+b\,i) = \\ &= [x^2-(x_1+x_2)x+x_1x_2][x^2-2ax+a^2+b^2] = \\ &= [x^2-(x_1+a+bi)x+(a+bi)x_1] \\ &\quad [x^2-(x_2+a-bi)x+(a-bi)x_2] = \\ &= [x^2-(x_1+a-bi)x+(a-bi)x_1] \\ &\quad [x^2-(x_2+a+bi)x+(a+bi)x_2]. \end{aligned} \tag{24}$$

Let $P_4(x)$ have only complex roots

$$\begin{aligned} P_4(x) &= (x-a-bi)(x-a+bi)(x-c-di)(x-c+di) = \\ &= [x^2-2ax+a^2+b^2][x^2-2cx+c^2+d^2] = \\ &= [x^2-(b+d)\,ix+ac-bd+(ad+bc)\,i] \\ &\quad [x^2+(b+d)\,ix+ac-bd-(ad+bc)\,i] = \\ &= [x^2-(b-d)\,ix+ac+bd+(bc-ad)\,i] \\ &\quad [x^2+(b-d)\,ix+ac+bd+(ad-bc)\,i]. \end{aligned} \tag{25}$$

Let $P_4(x)$ have only real roots

$$\begin{aligned} P_4(x) &= (x-x_1)(x-x_2)(x-x_3)(x-x_4) = \\ &= [x^2-(x_1+x_2)x+x_1x_2][x^2-(x_3+x_4)x+x_3x_4] = \\ &= [x^2-(x_1+x_3)x+x_1x_3][x^2-(x_2+x_4)x+x_2x_4] = \\ &= [x^2-(x_1+x_4)x+x_1x_4][x^2-(x_2+x_3)x+x_2x_3]. \end{aligned} \tag{26}$$

First, we shall prove the "only if" direction. Let $P_4(x)$ have two real and two complex roots. Then we have the first row in the factorizations (24) and the remaining three rows we get by considering all possibilities of factorizations with two quadratic polynomials having a unit as a leading coeficient (regardless of it having real coeficients or complex coeficients). These possibilities are closely connected with the roots of the Descartes's cubic resolvent $P_3(t)$, because $t=A^2$, where A is a coeficient of x in one of these two quadratic polynomials (no matter which one, because they differ only in the sign). As the sum of these two coeficients of x is the same in all those possibilities and de facto represents the coeficient of x^3 (which is zero), we get an important relation

$$x_1+x_2+2a=0. \tag{27}$$

Now, we shall consider two cases $x_1=x_2$ and $x_1\neq x_2$. In the first case, we obtain $x_1+a=x_2+a=0$, from (27) and finally, from this one and from (24), it follows

$$(A^2)_1=4a^2\geq 0;\ (A^2)_{2,3}=-b^2<0. \tag{28}$$

Thus $-b^2$ is a double root of $P_3(t)$ in the first case. In the second case from (27) we obtain

$$x_2+a=-(x_1+a). \tag{29}$$

From (29) we conclude that $x_1+a\neq 0$ and $x_2+a\neq 0$, because on the contrary, (29) leads to $x_1+a=x_2+a=0$ or equivalently to $x_1=x_2$, which is a contradiction. Thus, in the second case we obtain from (24)

$$(A^2)_{2,3}=(x_1+a)^2-b^2\pm 2b\,(x_1+a)\,i. \tag{30}$$

That means $(A^2)_{2,3}$ is a pair of conjugate complex numbers (because $b\neq 0$ and $x_1+a\neq 0$).

Now, let $P_4(x)$ have two pairs of conjugate complex numbers. It means that in (25) b and d are different from zero, which implies that

$$(b+d)^2\neq (b-d)^2. \tag{31}$$

From (25) and (31) we get easily

$$(A^2)_2=-(b+d)^2\neq (A^2)_3=-(b-d)^2. \tag{32}$$

Thus, $(A^2)_{2,3}$ are different nonpositive real numbers, and $(A^2)_1=4a^2$ is evidently a nonnegative real number.

Finally, let $P_4(x)$ have only real roots. From (26) it is easy to see that $(A^2)_{1,2,3}\geq 0$.

We shall furthermore prove the "if" direction. If we want to prove that the first case implies $P_4(x)$ having two real roots and a pair of conjugate complex roots we suppose the opposite, that the first case holds and for example $P_4(x)$ has two pairs of conjugate complex roots. We have proved before that if $P_4(x)$ has two pairs of conjugate complex roots, then it implies the second case. As the first and the second case are mutually exclusive cases, we come to a contradiction. The same type of a proof is valid if we suppose that the first case holds and $P_4(x)$ has four real roots. Hence, the exclusive property of the cases is the main tool in all remaining proofs.

Q.E.D.

Theorem 2.

$$\begin{aligned}
&\text{1st case} \iff\\
&\quad D_1>0\\
&\quad \text{or } (D_1=0 \text{ and } (a_2^2-4a_0<0 \text{ or } (a_2^2-4a_0>0 \text{ and } a_2>0)))\\
&\quad \text{or } (D_1=0 \text{ and } a_2^2-4a_0=0 \text{ and } a_2>0 \text{ and } a_1\neq 0)\\
&\text{2nd case} \iff\\
&\quad (D_1<0 \text{ and } (a_2^2-4a_0<0 \text{ or } (a_2^2-4a_0\geq 0 \text{ and } a_2>0)))\\
&\quad \text{or } (a_1=0 \text{ and } a_2^2-4a_0=0 \text{ and } a_2>0)\\
&\text{3rd case} \iff D_1\leq 0 \text{ and } a_2^2-4a_0\geq 0 \text{ and } a_2\leq 0.
\end{aligned} \tag{33}$$

Proof.

$$\begin{aligned}
P_3(t)&=t^3+2a_2t^2+(a_2^2-4a_0)\,t-a_1^2\\
P_3'(t)&=3t^2+4a_2t+a_2^2-4a_0\\
P_3''(t)&=6t+4a_2.
\end{aligned} \tag{34}$$

Since in the third case all roots are real (double and triple roots are included in that case) it is equivalent to $D_1\leq 0$ (see [3]), but all roots are not only real, but all roots are nonnegative, which is equivalent to $D_1\leq 0$ and both roots of $P_3'(t)$ are nonnegative. This last statement is equivalent to $D_1\leq 0$ and $P_3'(0)=a_2^2-4a_0\geq 0$ and $P_3''(0)=4a_2\leq 0$.

Since in the second case all roots are real and different (except in one special case which will be soon considered), which is equivalent to $D_1<0$, but as two roots are nonpositive and one nonnegative we conclude that either $P_3'(t)$ has one root negative and one root positive or both roots are nonpositive and different. The first case is equivalent to $D_1<0$ and $P_3'(0)=a_2^2-4a_0<0$. The second case is equivalent to $D_1<0$ and $P_3'(0)=a_2^2-4a_0\geq 0$ and $P_3''(0)=4a_2>0$. It remains only to consider the special case of the second case. In that special case one root of $P_3(t)$ is negative and two other roots are equal to zero. It is equivalent to $P_3(0)=-a_1^2=0$ (thus $a_1=0$) and one root of $P_3'(t)$ is negative while the other is zero. It is equivalent to $a_1=0$ and $P_3'(0)=a_2^2-4a_0=0$ and $P_3''(0)=4a_2>0$.

Since in the first case we have two quite different possibilities, we shall first consider the first possibility in which only one root of $P_3(t)$ is real and nonnegative. That first possibility is equivalent to $D_1>0$ (see [3]). In the second possibility we have one double negative real root and one nonnegative real root. It is equivalent to $D_1=0$ (see [3]) and at least one root of $P_3'(t)$ is negative. That is equivalent to $D_1=0$ and one root of $P_3'(t)$ is negative while the other is positive or both roots of $P_3'(t)$ are nonpositive and different. That means in the first case $D_1=0$ and $P_3'(0)=a_2^2-4a_0<0$ or in the second case $D_1=0$ and $P_3'(0)=a_2^2-4a_0\geq 0$ and $P_3''(0)=4a_2>0$. But we need to separate this second case additionally in two cases to make a distinction between it and a special case of the second case (see remark 1. and remark 2.).

Q.E.D.

Remark 1. It is easy to see that conditions $a_1=0$ and $a_2^2-4a_0=0$ imply $D_1=0$.

Remark 2. To make a distinction between the following two possibilities (see Figure 2.) we introduce these conditions in order to characterise the first case and the second case.

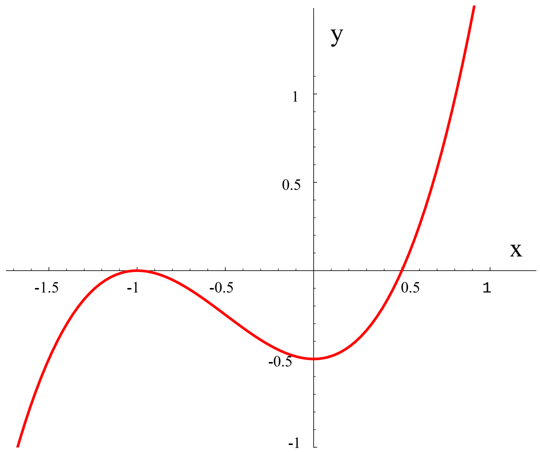

Fig. 2a: 1st case $D_1 = 0, a_1 \neq 0$, $P_3'(0) = a_2^2 - 4a_0 = 0$, $P_3''(0) = 4a_2 > 0$.

Remark 3. Ewerywhere in (33) the symbol ”or” is used only in the exclusive sense. Although the characterization of the first case and of the second case is quite complicated, their main parts are not so complicated (the main parts are those in which possibility $D_1 = 0$ is excluded). Hence, the main part of the first case is $D_1 > 0$ and the main part of the second case is $D_1 < 0$ and $(a_2^2 - 4a_0 < 0$ or $(a_2^2 - 4a_0 \geq 0$ and $a_2 > 0))$. These main parts, especially in the second case, are of considerable importance in the theory of plane quartic curves.

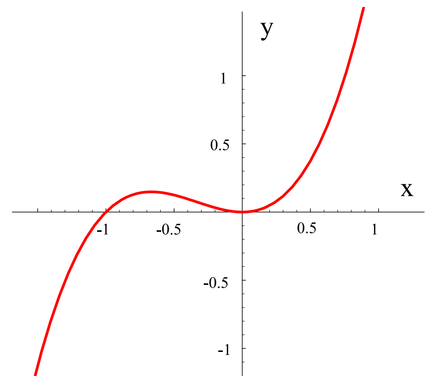

Fig. 2b: 2nd case $a_1 = 0$, $P_3'(0) = a_2^2 - 4a_0 = 0$, $P_3''(0) = 4a_2 > 0$.

References

[1] W. W. ROUSE BALL, *A short account of the history of mathematics*, Dover publications, New York, 1960.

[2] A. B. BASSET, *An elementary treatise on cubic and quartic curves*, Deighton Bell and Co., Cambridge, 1901.

[3] B. PAVKOVIĆ, D. VELJAN, *Elementarna matematika I*, Tehnička knjiga, Zagreb, 1992.

[4] J. PLEMELJ, *Algebra in teorija števil*, Slovenska akademija znanosti in umetnosti, Ljubljana, 1962.

[5] V. SEDMAK, *Uvod u algebru*, Sveučilište u Zagrebu, Zagreb, 1961.

Dr. sc. Radimir Viher
University of Zagreb
Faculty of Civil Engineering
e-mail: viher@grad.hr